SAARLAND UNIVERSITY
Department of Mathematics
Prof. Dr. Mark Groves
MSc Jens Horn

Mathematics for Computer Scientists 2, SS 2018
Sheet 9

---

1. For which values of λ and μ do the following real systems of linear equations have no solution, a unique solution, infinitely many solutions? Interpret your results geometrically.

(a)
$$\begin{aligned} 2x + 3y + z &= 5, \\ 3x - y + \lambda z &= 2, \\ x + 7y - 6z &= \mu \end{aligned}$$

(b)
$$\begin{aligned} x + y - 4z &= 0, \\ 2x + 3y + z &= 1, \\ 4x + 7y + \lambda z &= \mu \end{aligned}$$

2. Find all real nontrivial solutions of the equations

$$\begin{aligned} 2x_1 - 3x_2 - x_3 + x_4 &= 0, \\ 3x_1 + 4x_2 - 4x_3 - 3x_4 &= 0, \\ 17x_2 - 5x_3 - 9x_4 &= 0. \end{aligned}$$

Show that one of these solutions satisfies the equations

$$\begin{aligned} x_1 + x_2 + x_3 + x_4 + 1 &= 0, \\ x_1 - x_2 - x_3 - x_4 - 3 &= 0 \end{aligned}$$

but none can be written as a linear combination of the vectors $(0, 1, 2, 3)$ and $(3, 2, 1, 0)$.

3. Show that

$$\det \begin{pmatrix} a^2+1 & ab & ac \\ ab & b^2+1 & bc \\ ac & bc & c^2+1 \end{pmatrix} = a^2 + b^2 + c^2 + 1.$$

4. Let K be a field.

(a) Two matrices $A, B = K^{n\times n}$ are called *similar* if there is a matrix $P \in \mathrm{GL}(n, K)$ such that $B = P^{-1}AP$. Prove that

(i) similarity of matrices defines an equivalence relation $\sim$ on $K^{n\times n}$,

(ii) $A \sim B$ if and only if there is a finite-dimensional vector space V over K with bases $\mathcal{B}$, $\mathcal{B}'$ and a linear transformation $T : V \to V$ such that $A = M^{\mathcal{B}}_{\mathcal{B}}(T)$, $B = M^{\mathcal{B}'}_{\mathcal{B}'}(T)$.

(b) Prove that two similar matrices have the same determinant.

(c) Let V be an n-dimensional vector space over K. How would you define the determinant of a linear transformation $T : V \to V$?