SAARLAND UNIVERSITY
Department of Mathematics
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Mathematics for Computer Scientists 1, WS 2017/18
Sheet 9

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1. (a) Find all convergent subsequences of the sequence

$$1, -1, -1, 1, 1, 1, -1, -1, -1, -1, 1, 1, 1, 1, 1, \ldots.$$

(b) Find all convergent subsequences of the sequence

$$1, 2, 2, 1, 2, 2, 3, 3, 3, 1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots.$$

(c) For which real numbers α is there a subsequence of the sequence

$$\tfrac{1}{2}, \tfrac{1}{3}, \tfrac{2}{3}, \tfrac{1}{4}, \tfrac{2}{4}, \tfrac{3}{4}, \tfrac{1}{5}, \tfrac{2}{5}, \tfrac{3}{5}, \tfrac{4}{5}, \ldots,$$

which converges to α?

2. (a) Derive the formula

$$\sum_{k=1}^{n} \frac{1}{k(k+1)(k+2)} = \frac{1}{4} - \frac{1}{2(n+1)} + \frac{1}{2(n+2)}$$

[Hint:

$$\frac{1}{k(k+1)(k+2)} = \frac{1}{2k} - \frac{1}{k+1} + \frac{1}{2(k+2)}$$

for all $k \in \mathbb{N}$.]

(b) Prove that the series

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)(n+2)}$$

converges and determine its sum.

3. Consider the sequence $\{a_n\}$, where $a_n = (1 + \frac{1}{n})^n$.

(a) Show that

$$\frac{a_{n+1}}{a_n} = \left(1 - \frac{1}{(n+1)^2}\right)^{n+1} \frac{n+1}{n}, \qquad n \in \mathbb{N}.$$

(b) Use Bernoulli's inequality

$$(1+x)^k \geq 1 + x, \qquad x \geq -1,\ k \in \mathbb{N},$$

to prove the estimate

$$\frac{a_{n+1}}{a_n} \geq 1, \qquad n \in \mathbb{N}.$$

(c) Use the binomial expansion

$$(1+x)^n = \sum_{j=0}^{n} \frac{n!}{j!(n-j)!} x^j, \qquad |x| < 1,\ n \in \mathbb{N},$$

to prove the estimate

$$a_n \leq \sum_{j=0}^{n} \frac{1}{j!}, \qquad n \in \mathbb{N}.$$

[Hint:

$$\frac{n!}{(n-j)!} \frac{1}{n^j} = \frac{n}{n} \cdot \frac{n-1}{n} \cdot \ldots \cdot \frac{n-j+1}{n}$$

for $n \in \mathbb{N}$ and $j = 0, 1, \ldots, n$.]

(d) Prove that

$$a_n \leq 3, \qquad n \in \mathbb{N}.$$

[Hint: $2^{j-1} \leq j!$ for all $j \in \mathbb{N}$.]

(e) Deduce that $\{a_n\}$ converges to a real number in the interval $(2, 3)$. (This number is *Euler's number* e.)