SAARLAND UNIVERSITY
Department of Mathematics
Prof. Dr. Mark Groves
MSc Jens Horn

Mathematics for Computer Scientists 1, WS 2018/19
Sheet 7

---

1. Prove the following results using the pigeon-hole principle.

(a) In every collection of 7 integers there are at least two whose difference is divisible by 6.

(b) Let n be a natural number. In every collection of n^2+1 points $P_1, \ldots, P_{n^2+1}$ in a square of side length n there are at least two points separated by a distance of no more than $\sqrt{2}$.

(c) In every collection of 51 integers between 1 and 100 there are at least two whose sum is 101.

2. Prove that the set of all prime numbers is infinite. [Hint: Modify the proof that the set $\mathbb{N}$ is infinite. You may assume that a natural number $m \geq 2$ is either a prime number or divisible by a prime number.]

3. (a) Prove that the set of all finite subsets of $\mathbb{N}$ is countably infinite. [Hint: Arrange the subsets according to the sum of their elements.]

(b) Let A_1, A_2, A_3, ... be countably infinite sets. Prove that $\bigcup_{i=1}^{\infty} A_i$ is also countably infinite. [Hint: Denote the elements of A_i by $\{a_{i,1}, a_{i,2}, a_{i,3}, a_{i,4}, \ldots\}$ and apply a diagonal argument to count the elements $\{a_{i,j}\}_{i,j=1,2,\ldots}$ of $\bigcup_{i=1}^{\infty} A_i$.]

4. Determine the infima and suprema of the sets

$$M_0 = \left\{x \in \mathbb{Q} \ : \ \sqrt{3} < x \leq \sqrt{5}\right\},$$

$$M_1 = \left\{\frac{1}{m} + \frac{1}{n} \ : \ m, n \in \mathbb{Z} \setminus \{0\}\right\},$$

$$M_2 = \left\{\frac{x}{x+1} \ : \ x \in \mathbb{R}, \ x > 0\right\},$$

$$M_3 = \left\{\frac{x+1}{x} \ : \ x \in \mathbb{R}, \ x > 0\right\}$$

and decide whether their minima and maxima exist.