SAARLAND UNIVERSITY
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Mathematics for Computer Scientists 1, WS 2017/18
Sheet 8

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1. Prove the following statements.

(a) $\lim\limits_{n\to\infty} \dfrac{n}{n+1} = 1$

(b) $\lim\limits_{n\to\infty} \dfrac{n+3}{n^3+4} = 0$

(c) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[8]{n^2} = 0$

(d) $\lim\limits_{n\to\infty} \sqrt[8]{n^2+1} - \sqrt[4]{n+1} = 0$

(e) $\lim\limits_{n\to\infty} \dfrac{n!}{n^n} = 0$

[Hint: $\dfrac{2}{n}, \dfrac{3}{n}, \ldots \dfrac{n-1}{n} \leq 1$.]

(f) $\lim\limits_{n\to\infty} \sqrt[n]{n} = 1$

[Hint: For each $\varepsilon > 0$ one has that $\dfrac{n}{(1+\varepsilon)^n} \to 0$ as $n \to \infty$, and in particular there exists $N \in \mathbb{N}$ such that $\dfrac{n}{(1+\varepsilon)^n} < 1$ for all $n > N$.]

(g) $\lim\limits_{n\to\infty} \sqrt[n]{a} = 1$ für $a > 0$

[Hint: There exists $N \in \mathbb{N}$ with $\dfrac{1}{n} < a < n$ for all $n > N$.]

(h) $\lim\limits_{n\to\infty} \sqrt[n]{a^n+b^n} = a$ für $a > b > 0$

[Hint: $\sqrt[n]{a^n+b^n} = a\sqrt[n]{1+\left(\frac{b}{a}\right)^n}$ and $1 < 1+\left(\frac{b}{a}\right)^n < 2$.]

2. (a) Let X be a non-empty set of real numbers which is bounded above. Prove that there is a sequence $\{x_n\}$ of numbers in X which converges to $\sup X$. (The sequence $\{x_n\}$ is called a *maximising sequence* for X.)

[Hint: Use the final lemma in Section 2.6 with $\varepsilon = \frac{1}{n}$.]

Formulate a corresponding result for minimising sequences.

(b) Let r be an arbitrary real number. Prove that there exists a sequence $\{q_n\}$ of rational numbers which converges to r.

[Hint: Use the final remark in Section 2.7 with $\varepsilon = \frac{1}{n}$.]

3. The sequence $\{x_n\}$ is determined by the recursive scheme

$$x_1 = 2, \qquad x_{n+1} = 1 + \frac{6}{x_n}, \quad n = 1, 2, 3, \ldots .$$

Show that

$$\text{(i) } x_n \in [2,4] \;\Rightarrow\; x_{n+1} \in [2,4], \qquad \text{(ii) } x_{n+2} = 7 - \frac{36}{x_n + 6},$$

$$\text{(iii) } x_{n+2} \geq x_n \;\Rightarrow\; x_{n+3} \leq x_{n+1}, \qquad \text{(iv) } x_{n+2} \leq x_n \;\Rightarrow\; x_{n+3} \geq x_{n+1}$$

für $n = 1, 2, 3, \ldots$.

Prove that the sequences $x_1,\ x_3,\ x_5,\ \ldots$ and $x_2,\ x_4,\ x_6,\ \ldots$ converge and determine their limits. Deduce that $\{x_n\}$ converges and determine its limit.